SCHRÖDINGER'S INTERPOLATION PROBLEM AND ITS PROBABILISTIC SOLUTIONS

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Abstract: Probablistic solutions of the so called Schrödinger boundary data problem provide for a unique Markovian interpolation between any two strictly positive probability densities designed to form the input-output statistics data for a certain dynamical process taking place in a finite-time interval. The key problem is to select the jointly continuous in all variables positive semigroup kernel, appropriate for the phenomenological (physical) situation.

The issue of *deriving* a microscopic dynamics from the input-output statistics data (analyzed in terms of densities) was addressed, as the Schrödinger problem of a probabilistic interpolation, in a number of publications [1]– [6]. We shall consider Markovian propagation scenarios so remaining within the well established framework, where for any two Borel sets $A, B \subset R$ on which the strictly positive boundary densities $\rho(x,0)$ and $\rho(x,T)$ are defined, the transition probability $m(A,B)$ from the set A to the set B in the time interval $T > 0$ has a bi-variate density given in a specific factorized form: $m(x,y) = f(x)k(x,0,y,T)g(y)$, with marginals:

$$\int m(x,y)dy = \rho(x,0)\,,\ \int m(x,y)dx = \rho(y,T) \qquad (1)$$

Here, $f(x), g(y)$ are the a priori unknown functions, to come out as solutions of the integral system of equations (1), provided that in addition to the density boundary data we have in hands *any* strictly positive, continuous in space variables *function* $k(x,0,y,T)$. Additionally, we impose a restriction that $k(x,0,y,T)$ represents a certain strongly continuous dynamical semigroup kernel, while given at the time interval borders: it secures the Markov property of the sought for stochastic process.

It is the major mathematical discovery [2] that, without the semigroup assumption *but* with the prescribed, nonzero boundary data $\rho(x,0), \rho(y,T)$ *and* with the strictly positive continuous function $k(y,0,x,T)$, the system (1) of integral equations admits a unique solution in terms of two nonzero, locally integrable functions $f(x), g(y)$ of the same sign (positive, everything is up to a multiplicative constant).

If $k(y,0,x,T)$ is a particular, confined to the time interval endpoints, form of a concrete semigroup kernel $k(y,s,x,t), 0 \le s \le t < T$, then there exists a transition density:

$$p(y,s,x,t) = k(y,s,x,t)\frac{\theta(x,t)}{\theta(y,s)} \qquad (2)$$

defined in terms of functions

$$\theta(x,t) = \int dyk(x,t,y,T)g(y), \qquad \theta_*(y,s) = \int dxk(x,0,y,s)f(x) \tag{3}$$

which implements a consistent Markovian propagation of the probability density $\rho(x,t) = \theta(x,t)\theta_*(x,t)$ between its boundary versions, according to the standard transport recipe: $\rho(x,t) = \int p(y,s,x,t)\rho(y,s)dy$. For a given semigroup which is characterized by its generator (Hamiltonian), the kernel $k(y,s,x,t)$ and the emerging transition probability density $p(y,s,x,t)$ are unique in view of the uniqueness of solutions $f(x), g(y)$ of (1). For Markov processes, the knowledge of the transition probability density $p(y,s,x,t)$ for all intermediate times $0 \leq s < t \leq T$ suffices for the derivation of all other relevant characteristics.

In the framework of the Schrödinger problem the choice of the integral kernel $k(y,0,x,T)$ is arbitrary, except for the strict positivity and continuity demand. It is thus rather natural to ask for the most general stochastic interpolation, that is admitted under the above premises.

Clearly, the familiar strictly positive (Feynman-Kac) semigroup kernels generated by Laplacians plus suitable potentials are very special examples in a surprisingly rich encompassing family. Indeed, the concept of the "free noise", normally characterized by a Gaussian probability distribution appropriate to a Wiener process, can be extended to all infinitely divisible probability laws via the Lévy-Khintchine formula [7]. It expands our framework from continuous diffusion processes to jump or combined diffusion–jump propagation scenarios. All such (Lévy) processes are associated with strictly positive dynamical semigroup kernels, and all of them give rise to Markov solutions of the Schrödinger stochastic interpolation problem (1)-(3).

At this point, let us remark that apart from the wealth of physical phenomena described in terms of Gaussian stochastic processes, there is a number of physical problems where the Gaussian tool-box proves to be insufficient to provide satisfactory probabilistic explanations. Non–Gaussian Lévy processes naturally appear in the study of transient random walks when long-tailed distributions arise [8,9]. They are also found necessary to analyze fractal random walks [10], intermittency phenomena, anomalous diffusions, and turbulence at high Reynolds numbers [8,11]. On the other hand, our formulation of the Schrödinger interpolating dynamics can be regarded as a straightforward inversion of the well developed programme of studying dynamical systems (chaotic included) in terms of densities, [12,13].

Let us consider Hamiltonians (semigroup generators) of the form $H = F(\hat{p})$, where $\hat{p} = -i\nabla$ stands for the momentum operator and for $-\infty < k < +\infty$, $F = F(k)$ is a real valued, bounded from below, locally integrable function. Then, $exp(-tH) = \int_{-\infty}^{+\infty} exp[-tF(k)]dE(k)$, $t \geq 0$, where $dE(k)$ is the spectral measure of $\hat{p}$. We simplify further discussion by considering processes in one spatial dimension. Because $(E(k)f)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{k} exp(ipx)\hat{f}(p)dp$, where $\hat{f}$ is the Fourier transform of f, we learn that

$$[exp(-tH)]f(x) = [\int_{-\infty}^{+\infty} exp(-tF(k))dE(k)f](x) =$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} exp(-tF(k))exp(ikx)\hat{f}(k)dk = [exp(-tF(p))\hat{f}(p)]^{\vee}(x) \qquad (4)$$

where the superscript $\vee$ denotes the inverse Fourier transform.

Let us set $k_t = \frac{1}{\sqrt{2\pi}}[exp(-tF(p)]^{\vee}$, then the action of $exp(-tH)$ can be given in terms of a convolution: $exp(-tH)f = f * k_t$, where $(f * g)(x) := \int_R g(x - z)f(z)dz$.

We are interested in those $F(p)$ which give rise to positivity preserving semigroups: if $F(p)$ satisfies the celebrated Lévy-Khintchine formula, then k_t is a positive measure for all $t \geq 0$. The most general case refers to a contribution from three types of processes: deterministic, Gaussian, and an exclusively jump process. Let us concentrate on the integral part of the Lévy-Khintchine formula, which is responsible for arbitrary stochastic jump features:

$$F(p) = -\int_{-\infty}^{+\infty}[exp(ipy) - 1 - \frac{ipy}{1+y^2}]\nu(dy) \qquad (5)$$

where $\nu(dy)$ stands for the so-called Lévy measure. The disregarded Gaussian contribution would read $F(p) = p^2/2$; cf. Refs. [1-6] for an exhaustive discussion of related topics based on the concept of the conditional Wiener measure.

There are not many explicit examples (analytic formulas for probability densities involved) for processes governed by (5), except possibly for the so called stable probability laws. The best known example is the classic Cauchy density. Let us focus our attention on two selected choices for the characteristic exponent $F(p)$, namely: $F_0(p) = |p|$ which is the Cauchy process generator, and $F_m(p) = \sqrt{p^2+m^2} - m, m > 0$. Here, we have chosen suitable units so as to eliminate inessential parameters. The latter exponent is another form of the familiar classical relativistic Hamiltonian, better known as $\sqrt{m^2c^4 + c^2p^2} - mc^2$ where c is the velocity of light. The respective semigroup generators H_0, H_m are pseudodifferential operators. The associated kernels k_t^0, k_t^m in view of the "free noise" restriction (no potentials at the moment) are transition densities of the jump (Lévy) processes with intensities regulated by the corresponding Lévy measures $\nu_0(dy)$, $\nu_m(dy)$. The affiliated Markov processes solving the Schrödinger problem (1)-(3) immediately follow. It is instructive to notice that like in case of more traditional Gaussian derivations [4], the identities $\theta(x,t) \equiv 1$, $\theta_*(x,t) := \overline{\rho}(x,t)$ imply the pseudodifferential analog of the Fokker-Planck equation. It is a consequence of $[exp(-tH)\overline{\rho}](x) = \overline{\rho}(x,t)$ and of the identification $F(p \to -i\nabla) := H$. For example there holds: $F_0(p) \Longrightarrow \partial_t\overline{\rho}(x,t) = -|\nabla|\overline{\rho}(x,t)$. This evolution rule gives rise to the Cauchy process with its long-tailed probability density $rho(x,t) = \frac{1}{\pi}\frac{t}{t^2+x^2}$ and the transition probability density (e.g. the semigroup kernel in this free propagation case) of the same functional form with $x \to x - y$ and $t \to t - s$. Let us emphasize that the existence and uniqueness of solutions proof for the Schrödinger problem extends to all cases governed by the infinitely divisible probability laws, and can be generalized to encompass the additive perturbations by physical potentials (in analogy with the familiar Feynman-Kac formula).

Our semigroups are holomorphic, hence we can replace the time parameter t by a complex one $\sigma = t + is$, $t > 0$ so that $exp(-\sigma H) = \int_R exp(-\sigma F(k))\, dE(k)$. Its action is defined by $[exp(-\sigma H)]f = [(\hat{f} exp(-\sigma F)]^\vee = f * k_\sigma$. Here, the kernel reads $k_\sigma = \frac{1}{\sqrt{2\pi}}[exp(-\sigma F)]^\vee$. Since H is selfadjoint, the limit $t \downarrow 0$ leaves us with the unitary group $exp(-isH)$, acting in the same way: $[exp(-isH)]f = [\hat{f} exp(-isF)]^\vee$, except that now $k_{is} := \frac{1}{\sqrt{2\pi}}[exp(-isF)]^\vee$ in general is *not* a measure. In view of unitarity, the unit ball in L^2 is an invariant of the dynamics. Hence density measures can be associated with solutions of the Schrödinger pseudodiferential equations: $F_0(p) \Longrightarrow i\partial_t \psi(x,t) = |\nabla|\psi(x,t)$ or $F_m(p) \Longrightarrow i\partial_t\psi(x,t) = [\sqrt{-\triangle + m^2} - m]\psi(x,t)$, if provided with the appropriate initial data functions $\psi(x,0)$. Let us point out that we know in detail how the analytic continuation in time of the Laplacian generated holomorpic semigroup induces a mapping to diffusion processes of a quantum mechanical proveninence (since the standard Schrödinger equation $i\partial_t\psi(x,t) = -\triangle\psi(x,t)$ is involved).

All that ultimately submits the unitary (quantum) Schrödinger picture dynamics, with quite a variety of admissible semigroup generators to be used instead of the traditional Laplacian, to the stochastic analysis in the framework of the Schrödinger (again) boundary data problem. The natural question to be answered is: what are the stochastic processes consistent with the probability measure dynamics $\rho(x,t) = |\psi(x,t)|^2$ determined by pseudodifferential equations, eventually in the presence of external force fields ? The answer to this and related questions of the more pedestrian, nonequlibrium statistical physics origin can be found elswhere, [6,13,14].

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